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Soliton Evolution in the Composite-Boson Field

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We construct an elementary topological current in composite-boson field and point out that the time component of this topological current is the density of solitons in the field. Based on the implicit function theorem and the Taylor expansion, the evolution of solitons is detailed in the neighborhoods of the branch points of the composite-boson field. We also find that solitons generate or annihilate at the limit points and encounter one another, split, or merge at the bifurcation points of the composite-bosons field.

1. INTRODUCTION

An electron may turn into a boson by forming a flux-charge composite in an external magnetic field, which is called a composite boson. As a result electrons may condense without making Cooper pairs. When Bose condensation occurs, the two-dimensional electron gas becomes an incompressible fluid, which is the fractional quantum Hall (QH) state [1]. The kinematics and the dynamics of quantum coherence based on an improved composite-boson (CB) theory was studied by Ezawa [2]. Study of the topological structure of this kind of CB field was shown to be necessary. In this paper, we analyze the inner structure of the topological current and corresponding topological charge (winding number) of the CB field and show that the time component of the topological current is just the density of solitons in the CB field. Furthermore, in terms of the ϕ-mapping method [3] we introduce the criterion of soliton generation and annihilation, that is, vanishing of the mapping Jacobian. This condition defines the bifurcation points and allows us to consider the processes of merging and splitting of solitons in a small vicinity of the bifurcation points. All the results in this paper are obtained

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only from the viewpoint of topology without using any particular model or hypothesis.

This paper is organized as follows. In Section 2 we start with the CB field and construct an elementary topological current. By means of ϕ-mapping topological current theory, we prove that the time component of the topological current is the density of the solitons in the CB field. In Section 3 several crucial cases of branch processes are discussed and solitons are found to generate or annihilate at the limit points, which shows that the soliton system is unstable at these branch points. In Section 4 we consider the bifurcation of the soliton velocity field, and show that solitons encounter one another, split, or merge at the bifurcation points. The velocity of solitons is infinite when they are annihilating or generating, which is obtained only from the topological properties of the CB field. The topological charge conservation near the branch points is also studied.

2. TOPOLOGICAL STRUCTURE OF COMPOSITE-BOSON FIELD

The composite-boson picture is very useful for understanding the quantum Hall ferromagnet [2]. For the simplest CB field

$$\phi(\vec{x}) = e^{i\chi(\vec{x})}\sqrt{\sigma(x)} = e^{i\chi(\vec{x})}\sqrt{\rho_0 + \rho(\vec{x})} \tag{1}$$

where σ is the electron density, ρ_0 is the average density, $\rho(x)$ is the density fluctuation, and $\chi(x)$ is the conjugate phase. The field can be denoted by

$$\phi = \phi^1 + i\phi^2 \tag{2}$$

and we can regard ϕ as the complex representation of a two-dimensional vector field

$$\vec{\phi} = (\phi^1, \phi^2) \tag{3}$$

Then we define the two-dimensional unit vector field of the CB field,

$$n^a = \phi^a/\|\phi\|, \qquad \|\phi\|^2 = \phi^a\phi^a \tag{4}$$

satisfying

$$n^a n^a = 1, \qquad a = 1, 2 \tag{5}$$

From Eq. (4), it is easy to see that the zeros of the field $\vec{\phi}$ are just the singularities of $\vec{n}$. Then we can construct a topological current of the CB field in the (2 + 1)-dimensional space-time R^{2+1} with coordinates $x^1 = x$, $x^2 = y$, and $x^0 = t$,

$$Q_V^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab}\, \partial_j n^a\, \partial_k n^b, \qquad i, j, k = 0, 1, 2 \tag{6}$$

Obviously, the current (6) is identically conserved,

$$\partial_i Q_V^i = 0 \tag{7}$$

Following the ϕ-mapping topological current theory it can be rigorously proved that

$$Q_V^i = \delta^2(\vec{\phi})\, D^i\left(\frac{\phi}{x}\right) \tag{8}$$

where the Jacobian $D^i(\phi/x)$ is defined as

$$\epsilon^{ab} D^i\left(\frac{\phi}{x}\right) = \epsilon^{ijk}\, \partial_i \phi^a\, \partial_j \phi^b$$

in which the usual two-dimensional Jacobian is

$$D\left(\frac{\phi}{x}\right) = D^0\left(\frac{\phi}{x}\right)$$

The topological charge density

$$Q_V^0(\vec{x}) = \frac{1}{2\pi} \epsilon^{\mu\nu}\, \epsilon_{ab}\, \partial_\mu n^a \partial_\nu n^b, \qquad \mu, \nu = 1, 2 \tag{9}$$

is just the time component of the topological current. We will show that this topological charge density is just the density of solitons defined in the CB field. First, as suggested in ref. 8, we define the dressed CB field $\psi(\vec{x})$ by

$$\psi(x) = e^{-\tilde{A}(\vec{x})}\, \phi(\vec{x})$$

where $\tilde{A}(\vec{x})$ is the effective vector potential, which satisfies

$$eB_{\text{eff}} = \nabla^2 \tilde{A}(\vec{x}) = 2\pi m \rho(\vec{x}), \qquad \nu = \frac{1}{m} \tag{10}$$

Here B_{eff} [8, 9] is the effective magnetic field and ν is the Landau filling factor. From Eqs. (1) and (4), we obtain that

$$\nabla^2(\tilde{A}(\vec{x}) - \ln\sqrt{\rho_0 + \rho(\vec{x})}) = -\epsilon_{\mu\nu}\, \epsilon^{ab}\, \partial_\mu n^a\, \partial_\nu n^b \tag{11}$$

According to (10), one can further get

$$\frac{\nu}{4\pi}\nabla^2 \ln\left(1+\frac{\rho(\vec{x})}{\rho_0}\right)-\rho(\vec{x})=\nu\frac{1}{2\pi}\epsilon_{\mu\nu}\epsilon^{ab}\partial_\mu n^a\partial_\nu n^b \qquad (12)$$

The Cauchy–Riemann condition yields a differential equation for the density modulation [2],

$$\frac{\nu}{4\pi}\nabla^2 \ln\left(1+\frac{\rho(\vec{x})}{\rho_0}\right)-\rho(\vec{x})=\nu Q_{\text{top}}(\vec{x}) \qquad (13)$$

Comparing Eqs. (12) and (13), one can find that the soliton density Q_{top} is just the topological charge density Q_V^0.

From Eq. (8), one can get

$$Q_{\text{top}}(\vec{x})=\delta^2(\vec{\phi})\,D\left(\frac{\phi}{x}\right) \qquad (14)$$

which shows that $Q_{\text{top}}(\vec{x})$ does not vanish at the zero points of $\vec{\phi}$, i.e.,

$$\begin{aligned}\phi^1(x^1,x^2,t)&=0\\ \phi^2(x^1,x^2,t)&=0\end{aligned} \qquad (15)$$

determine the locations of zero. If the Jacobian determinant is given as

$$D\left(\frac{\phi}{x}\right)=\frac{\partial(\phi^1,\phi^2)}{\partial(x^1,x^2)}\neq 0$$

then the solutions of Eq. (15) are expressed as

$$x^1=x_l^1(t),\qquad x^2=x_l^2(t),\qquad l=1,2,\ldots,N \qquad (16)$$

which are the worldlines of N solitons $\vec{r}_l(t)$ ($l=1,2,\ldots,N$), of which the lth soliton is charged with the topological charge $\beta_l\eta_l$.

According to the δ-function theory [10] and the φ-mapping topological current theory, one can prove that

$$\delta^2(\vec{\phi})=\sum_{l=1}^{N}\frac{\beta_l}{|D(\phi/x)_{\vec{r}_l}|}\delta^2(\vec{x}-\vec{r}_l) \qquad (17)$$

where the positive integer β_l is called the Hopf index [5, 6] of map $x\to\phi$. The meaning of β_l is that when the point $\vec{x}$ covers the neighborhood of the zero $\vec{x}_l$ once, the vector field $\vec{\phi}$ covers the corresponding region β_l times. Using this expansion of $\delta^2(\vec{\phi})$ in (17), we see that

$$\delta^2(\vec{\phi})\,D\left(\frac{\phi}{x}\right)=\sum_{l=1}^{N}\beta_l\eta_l\delta^2(\vec{x}-\vec{r}_l) \qquad (18)$$

where η_l is the Brouwer degree [5, 6]:

$$\eta_l = \frac{D(\phi/x)}{|D(\phi/x)|}\bigg|_{\vec{r}_l} = \pm 1 \tag{19}$$

Direct substitution of (18) into (12) and (13) leads to the density of solitons in the following form:

$$Q_{\text{top}}(\vec{x}) = \sum_{l=1}^{N} \beta_l \eta_l \delta^2(\vec{x} - \vec{r}_l) \tag{20}$$

Here one can see that the density of solitons (14) and (20) is obtained directly from the definition of the topological charge of the CB field, which is more general than usually considered.

Following our theory, one can also get the velocity of the lth soliton,

$$V_l^i = \frac{dx_l^i}{dt} = \frac{D^i(\phi/x)}{D(\phi/x)}\bigg|_{\vec{r}_l}, \qquad i = 1, 2$$

from which one can identify the soliton velocity field as

$$V^i = \frac{D^i(\phi/x)}{D(\phi/x)}, \qquad i = 1, 2 \tag{21}$$

The expressions given by Eq. (21) for the velocity of solitons are useful because they avoid the problem of having to specify the position of solitons explicitly. The positions are implicitly determined by the zero of the CB field.

The current density of solitons (N solitons with topological charge $\beta_l \eta_l$ moving in space) can be written as the same form as the current density in hydrodynamics:

$$J^i = \sum_{l=1}^{N} \beta_l \eta_l \delta^2(\vec{r} - \vec{r}_l(t)) \frac{dx_l^i}{dt}$$

From Eqs. (8) and (17), the current density of solitons can be written in the concise form

$$J^i = Q_V^i = \delta^2(\vec{\phi}) D^i(\phi/x)$$

or

$$J^i = \epsilon^{i\nu\lambda} \epsilon_{ab} \partial_\nu n^a \partial_\lambda n^b$$

According to Eq. (7), the topological charges of the solitons are conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

which is only the topological property of the CB field.

The above discussions are based on the condition that the Jacobian is given as

$$D\left(\frac{\phi}{x}\right) \neq 0 \tag{22}$$

When the condition (22) fails, the solutions (16) of Eq. (15) will change in some way. It is interesting to discuss what will happen and what it corresponds to physically.

3. THE GENERATION AND ANNIHILATION OF SOLITONS

When

$$D\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{z}_l)} = 0$$

there exists the crucial case of branch processes. There are two kinds of branch points, limit points and bifurcation points.

First we study the case when the condition (22) fails and $D^1(\phi/x) \neq 0$. The usual implicit function theorem is of no use when $D(\phi/x) = 0$. Thus, to use the implicit function theorem to study the branch properties of solitons, we use the Jacobian $D^1(\phi/x)$ instead of the $D^3(\phi/x)$ to search for the solutions of $\vec{\phi}(x) = 0$. This means we have replaced t by x^1. Then we have a unique solution of Eqs. (15) in the neighborhood of the points $(t^*, \vec{z}_l)$,

$$\begin{aligned} t &= t(x^1) \\ x^2 &= x^2(x^1) \end{aligned} \tag{23}$$

with $t^* = t(z_l^1)$. We call the critical points $(t^*, \vec{z}_l)$ the limit points. In the present case,

$$\frac{dx^1}{dt}\bigg|_{(t^*,\vec{z}_l)} = \frac{D^1(\phi/x)}{D(\phi/x)}\bigg|_{(t^*,\vec{z}_l)} = \infty \tag{24}$$

i.e.,

$$\frac{dt}{dx^1}\bigg|_{(t^*,\vec{z}_l)} = 0 \tag{25}$$

The Taylor expansion of the solution of Eq. (23) at the limit point $(t^*, \vec{z}_l)$ is [3]

$$t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2}\bigg|_{(t^*, \vec{z}_l)} (x^1 - z_l^1)^2 \tag{26}$$

which is a parabola in the x^1–t plane. From (26) we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give the branch solutions of solitons at the limit points. If

$$\frac{d^2 t}{(dx^1)^2}\bigg|_{(t^*, \vec{z}_l)} > 0$$

we have the branch solutions for $t > t^*$ (see Fig. 1a), otherwise we have the branch solutions for $t < t^*$ (see Fig. 1b). These two cases are related to the origin and annihilation of solitons. From Eq. (24), one can also find the result that the velocity of solitons is infinite when they are generating or annihilating, which is found only from the topology of the CB field.

Since the topological current of the CB field is identically conserved, the topological charges of these two generated or annihilated solitons must be opposite at the limit point, i.e.,

$$\beta_{l_1}\eta_{l_1} = -\beta_{l_2}\eta_2$$

which shows that $\beta_{l_1} = \beta_{l_2}$ and $\eta_{l_1} = -\eta_{l_2}$.

For a limit point, one also requires

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*, \vec{z}_l)} \neq 0$$

As to a bifurcation point [12], it must satisfy a more complex condition. This case will be discussed in the following section in detail.

4. BIFURCATION OF SOLITONS

Now let us turn to consider the other case, in which the restrictions of Eqs. (15) at the bifurcation point $(t^*, \vec{z}_l)$ are

$$D\left(\frac{\phi}{x}\right)\bigg|_{(t^*, \vec{z}_l)} = 0, \qquad D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*, \vec{z}_l)} = 0 \tag{27}$$

These two restrictive conditions lead to the important fact that the functional relationship between t and x^1 is not unique in the neighborhood of the bifurcation point $(t^*, \vec{z}_l)$. The equation

$$\frac{dx^1}{dt}\bigg|_{(t^*, \vec{z}_l)} = \frac{D^1(\phi/x)}{D(\phi/x)}\bigg|_{(t^*, \vec{z}_l)} \tag{28}$$

under the restraint (27) directly shows that the direction of the integral curve

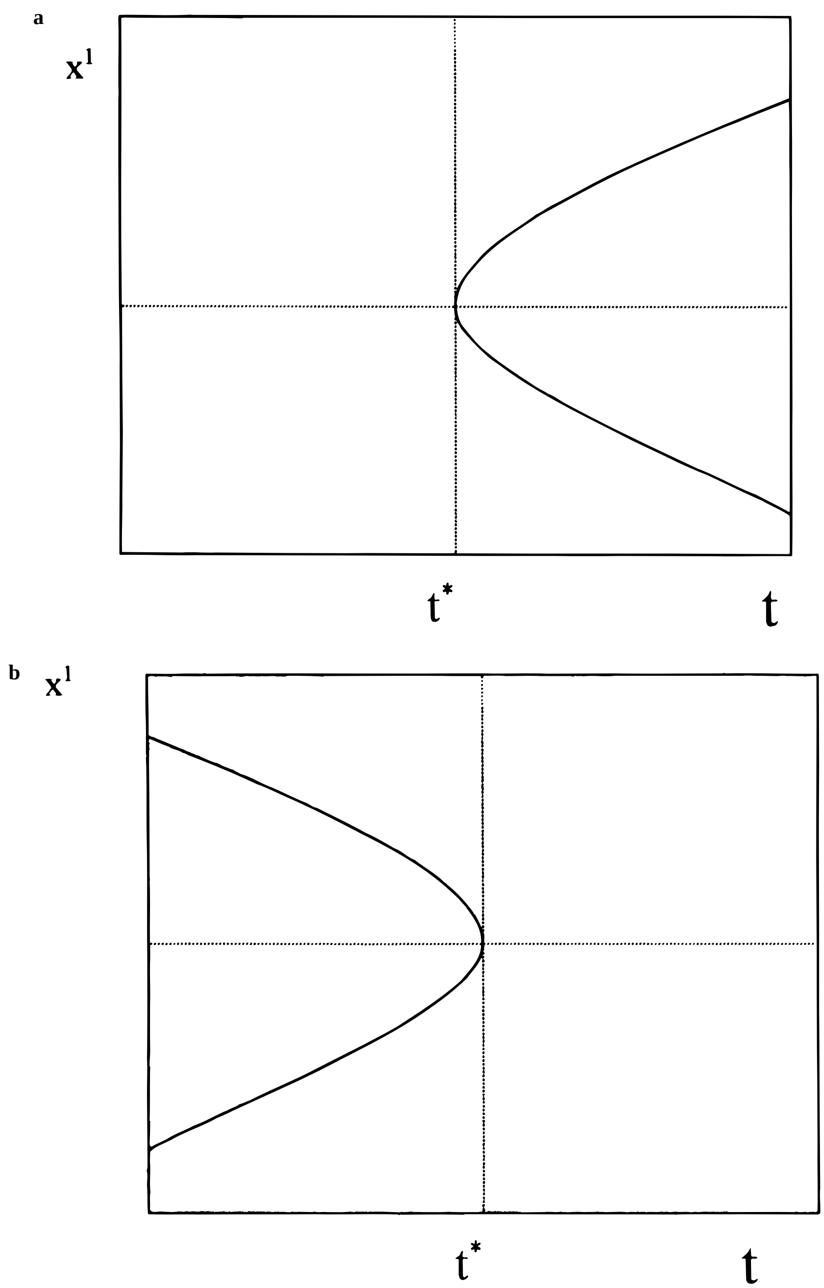

Fig. 1. Projecting the worldlines of solitons onto the x^1–t plane. (a) A pair of solitons with opposite charges generate at the limit point, i.e., the origin of solitons. (b) A pair of solitons with opposite charges annihilate at the limit point.

of Eq. (28) is indefinite, i.e., the velocity field of solitons is indefinite at the point $(t^*, \vec{z}_l)$. This is why the point $(t^*, \vec{z}_l)$ is called a bifurcation point of the CB field $\vec{\phi}$. To find the different directions of all branch curves at the bifurcation point, we suppose that

$$\left.\frac{\partial \phi^1}{\partial x^2}\right|_{(t^*,\vec{z}_l)} \neq 0 \tag{29}$$

From $\phi^1(x^1, x^2, t) = 0$, the implicit function theorem says that there exists one and only one functional relationship

$$x^2 = x^2(x^1, t) \tag{30}$$

According to the ϕ-mapping topological current theory, the Taylor expansion of the solution of Eqs. (15) in the neighborhood of the bifurcation point $(t^*, \vec{z}_l)$ can be expressed as [3]

$$A(x^1 - x_l^1)^2 + 2B(x^1 - x_l^1)(t - t^*) + C(t - t^*)^2 = 0$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\,\frac{dx^1}{dt} + C = 0 \tag{31}$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\,\frac{dt}{dx^1} + A = 0 \tag{32}$$

where A, B, and C are three constants. The solutions of Eq. (31) or Eq. (32) give different directions of the branch curves (worldlines of solitons) at the bifurcation point. There are four possible cases, which will show the physical meaning of the bifurcation points.

Case 1 ($A \neq 0$). For $\Delta = 4(B^2 - AC) > 0$, from Eq. (31) we get two different directions of the velocity field of solitons,

$$\left.\frac{dx^1}{dt}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \tag{33}$$

which is shown in Fig. 2, where two worldlines of solitons intersect with different directions at the bifurcation. This shows that two solitons encounter one another and then depart at the bifurcation point.

Case 2 ($A \neq 0$). For $\Delta = 4(B^2 - AC) = 0$, from Eq. (31) we get only one direction of the velocity field of solitons,

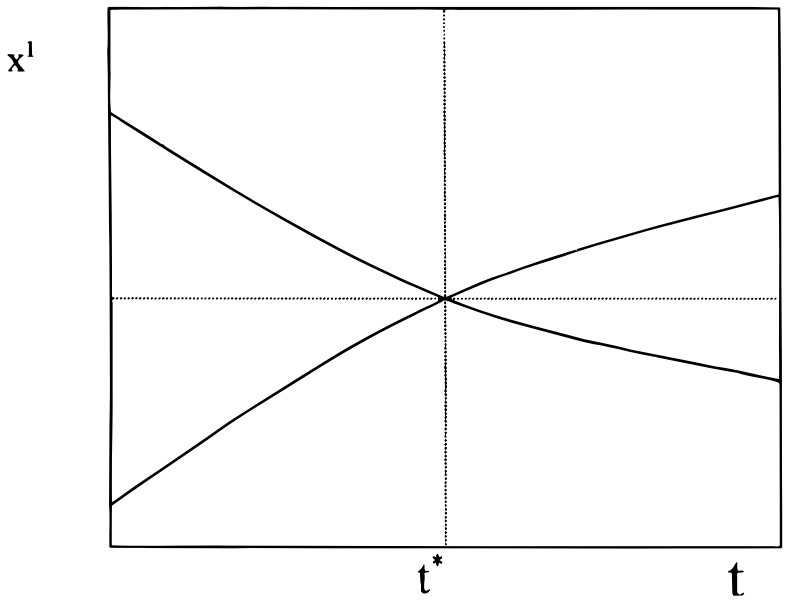

Fig. 2. Projecting the worldlines of solitons onto the x^1–t plane. Two worldlines intersect with different directions at the bifurcation point, i.e., two solitons encounter one another at the bifurcation point.

$$\left.\frac{dx^1}{dt}\right|_{1,2} = -\frac{B}{A} \tag{34}$$

which includes three important cases. First, two worldlines tangentially contact, i.e., two solitons tangentially encounter one another at the bifurcation point (see Fig. 3a). Second, two worldlines merge into one worldline, i.e., two solitons merge into one soliton at the bifurcation point (see Fig. 3b). Finally, one worldline resolves into two worldlines, i.e., one soliton splits into two solitons at the bifurcation point (see Fig. 3c).

Case 3 ($A = 0, C \neq 0$). For $\Delta = 4(B^2 - AC) = 0$, from Eq. (32) we have

$$\left.\frac{dt}{dx^1}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, \quad -\frac{2B}{C} \tag{35}$$

There are two important cases: first, one worldline resolves into three worldlines, i.e., one soliton splits into three solitons at the bifurcation point (see Fig. 4a). Second, three worldlines merge into one worldline, i.e., three solitons merge into one soliton at the bifurcation point (see Fig. 4b).

Case 4 ($A = C = 0$). Equations (31) and (32) give, respectively,

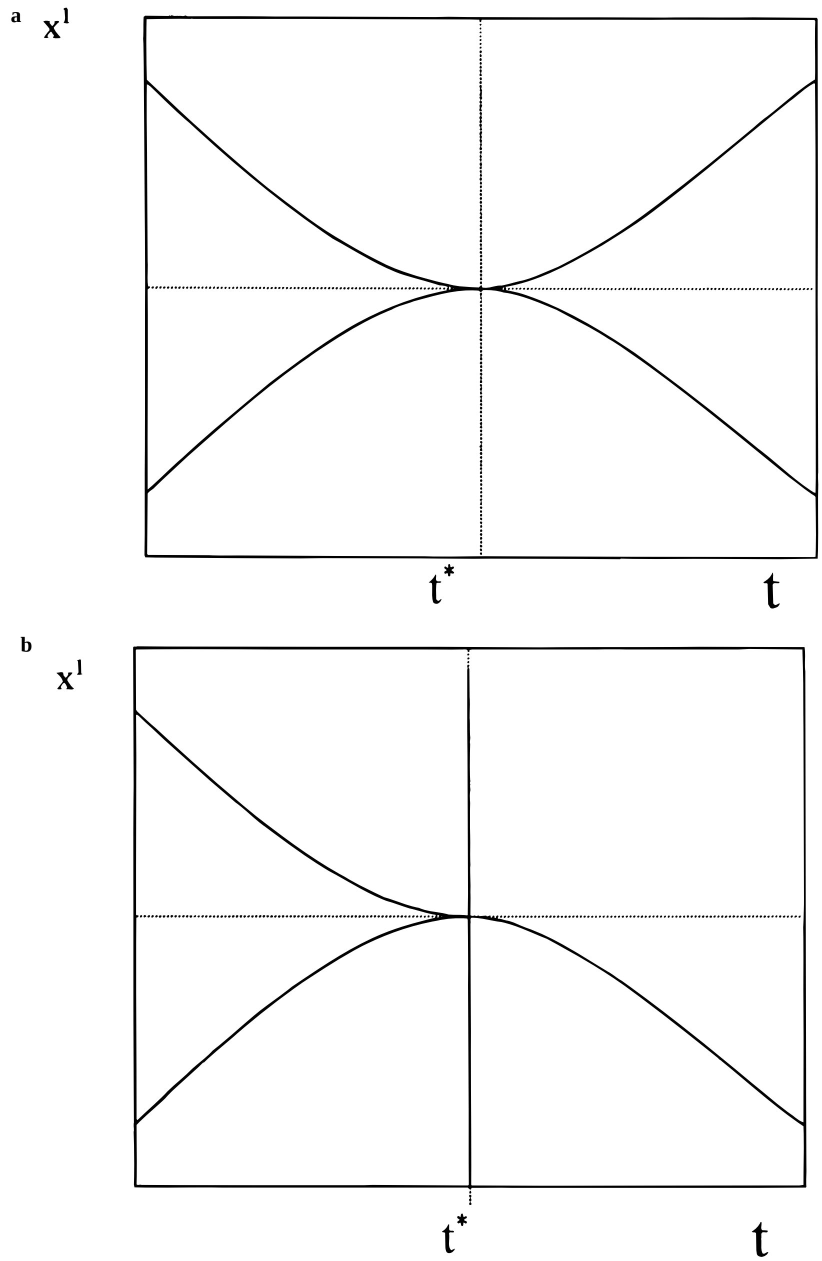

Fig. 3. (a) Two worldlines tangentially contact, i.e., two solitons tangentially encounter one another at the bifurcation point. (b) Two worldlines merge into one worldline, i.e., two solitons merge into one soliton at the bifurcation point. (c) One worldline resolves into two worldlines, i.e., one soliton splits into two solitons at the bifurcation point.

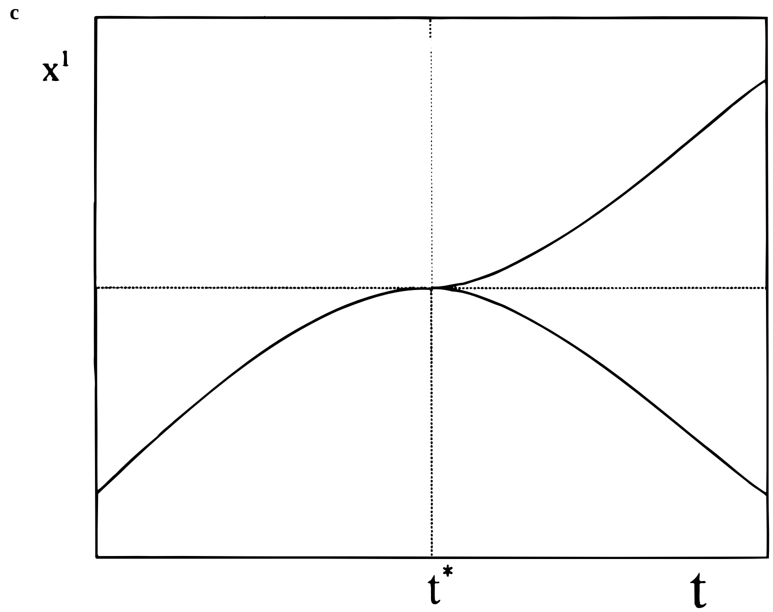

Fig. 3. Continued.

$$\frac{dx^1}{dt} = 0, \qquad \frac{dt}{dx^1} = 0 \tag{36}$$

This case is obvious (Fig. 5) and is similar to Case 3.

The above solutions reveal the evolution of solitons. Besides the encounter of solitons, i.e., two solitons encountering one another at the bifurcation point (see Figs. 2 and Fig. 3a), it also includes the splitting and merging of solitons. When a multicharged soliton moves through the bifurcation point, it may split into several solitons along different branch curves (see Figs. 3c, 4a, and 5b). In contrast, several solitons can merge into one soliton at the bifurcation point (see Figs. 3c, 4b, and 5a). The identical conservation of the topological charge shows that the sum of the topological charge of the final soliton (solitons) must be equal to that of the initial soliton (solitons) at the bifurcation point, i.e.,

$$\sum_f \beta_{l_f}\eta_{l_f} = \sum_i \beta_{l_i}\eta_{l_i}$$

for fixed l. Furthermore, the generation, annihilation, and bifurcation of solitons are not gradual changes, but start at a critical value of arguments, i.e., they are sudden changes.

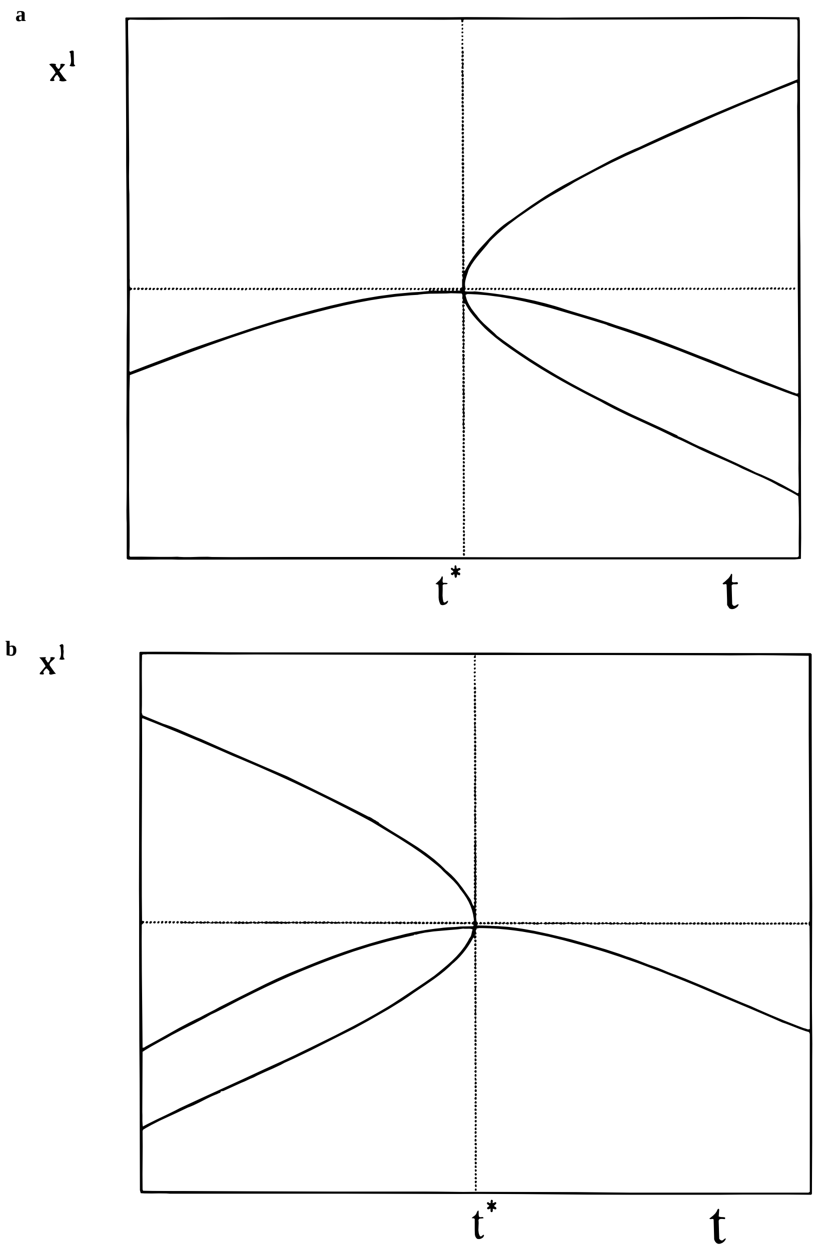

Fig. 4. Two important cases of Eq. (35). (a) One worldline resolves into three worldlines, i.e., one soliton splits into three solitons at the bifurcation point. (b) Three worldlines merge into one worldline, i.e., three solitons merge into one soliton at the bifurcation point.

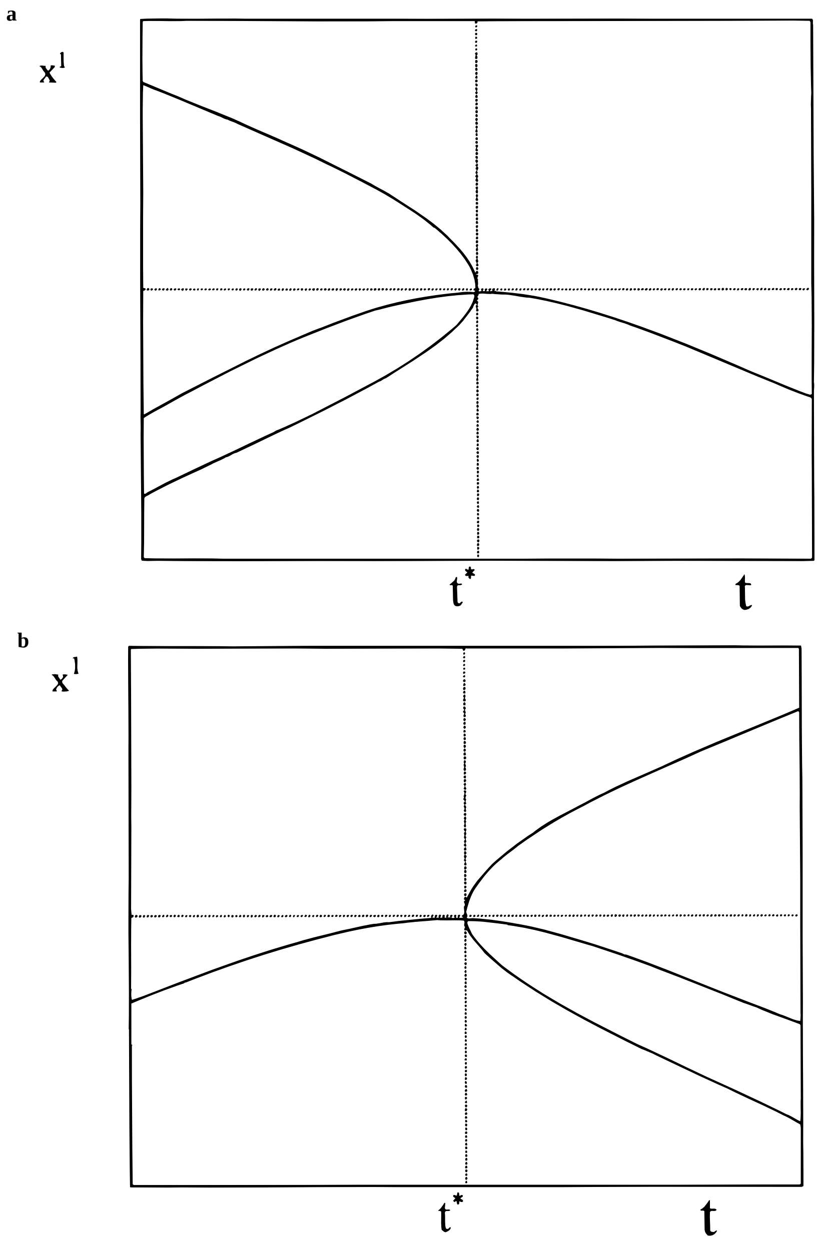

Fig. 5. Two worldlines intersect normally at the bifurcation point. This case is similar to Fig. 4. (a) Three solitons merge into one soliton at the bifurcation point. (b) One soliton splits into three solitons at the bifurcation point.

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